Published by Institute of Physics Publishing for SISSA

Received: *May 1, 2008*
Accepted: *May 18, 2008*
Published: *May 28, 2008*

Partition functions of torsion > 1 dyons in heterotic string theory on T^6

Shamik Banerjee, Ashoke Sen and Yogesh K. Srivastava

Harish-Chandra Research Institute,
Chhatnag Road, Jhusi, Allahabad 211019, India
E-mail: bshamik@mri.ernet.in, sen@mri.ernet.in, yogesh@mri.ernet.in

Abstract: The original proposal of Dijkgraaf, Verlinde and Verlinde for the quarter BPS dyon partition function in heterotic string theory on T^6 is known to correctly produce the degeneracy of dyons of torsion 1, i.e. dyons for which $\gcd(Q \wedge P) = 1$. We propose a generalization of this formula for dyons of arbitrary torsion. Our proposal satisfies the constraints coming from S-duality invariance, wall crossing formula, black hole entropy and the gauge theory limit. Furthermore using our proposal we derive a general wall crossing formula that is valid even when both the decay products are non-primitive half-BPS dyons.

Keywords: Black Holes in String Theory, String Duality.

JHEP05(2008)098

Contents

1. Introduction

Since the original proposal of Dijkgraaf, Verlinde and Verlinde [1] for quarter BPS dyon spectrum in heterotic string theory compactified on T^6, there has been extensive study of dyon spectrum in a variety of $\mathcal{N}=4$ supersymmetric string theories [2 − 18] and also in $\mathcal{N}=8$ and $\mathcal{N}=2$ supersymmetric string theories [19, 20]. However it has been realised for some time that even in heterotic string theory on T^6 the proposal of [1] gives the correct dyon spectrum only for a subset of dyons, — those with unit torsion, i.e. for which the electric and magnetic charge vectors Q and P satisfy gcd$(Q \wedge P)$=1 [14, 21, 22]. In a previous paper we proposed a general set of constraints which must be satisfied by the partition function of quarter BPS dyons in any $\mathcal{N}=4$ supersymmetric string theory and used these constraints to propose a candidate for the dyon partition function for torsion two dyons in heterotic string theory on T^6 [23]. In this paper we extend our analysis to dyons of arbitrary torsion and propose a form of the partition function of such dyons.

2. Proposal for the partition function

We consider the set $\mathcal{B}$ of all dyons of charge vectors (Q,P) in heterotic string theory on T^6, with Q being r times a primitive vector, P a primitive vector and Q/r and P admitting a primitive embedding in the Narain lattice [24, 25], i.e. all lattice vectors lying in the plane of Q and P can be expressed as integer linear combinations of Q/r and P. These dyons have torsion r, i.e. gcd$(Q \wedge P) = r$. It was shown in [21, 22] that given any pair (Q,P) of this type with the same values of Q^2, P^2 and $Q\cdot P$, they are related by T-duality transformation. We denote by $d(Q,P)$ the index measuring the number of bosonic supermultiplets minus the number of fermionic supermultiplets of quarter BPS dyons carrying charges (Q,P) — up to a normalization this can be identified with the helicity supertrace B_6 introduced

JHEP05(2008)098

in [26]. T-duality invariance of the theory tells us that $d(Q,P)$ must be a function of the T-duality invariants, and hence has the form $f(Q^2, P^2, Q\cdot P)$. Then the dyon partition function $1/\check\Phi(\check\rho,\check\sigma,\check v)$ is defined as

$$\frac{1}{\check\Phi(\check\rho,\check\sigma,\check v)} = \sum_{Q^2,P^2,Q\cdot P} (-1)^{Q\cdot P+1} f(Q^2,P^2,Q\cdot P)\, e^{i\pi(\check\sigma Q^2+\check\rho P^2+2\check v Q\cdot P)} . \tag{2.1}$$

The sum in (2.1) runs over all possible values of Q^2, P^2 and $Q\cdot P$ in the set $\mathcal{B}$. This in particular requires

$$Q^2/2 \in r^2\mathbb{Z}, \qquad P^2/2\in\mathbb{Z}, \qquad Q\cdot P\in r\mathbb{Z} . \tag{2.2}$$

The imaginary parts of $(\check\rho,\check\sigma,\check v)$ in (2.1) need to be adjusted to lie in a region where the sum is convergent. Although the index f and hence the partition function $1/\check\Phi$ so defined could depend on the domain of the asymptotic moduli space of the theory in which we are computing the partition function [13, 18, 22, 23], in all known examples the dependence of $\check\Phi$ on the domain is found to come through the region of the complex $(\check\rho,\check\sigma,\check v)$ plane in which the sum is convergent. Thus (2.1) computed in different domains in the asymptotic moduli space of the theory describes the same analytic function $\check\Phi$ in different domains in the complex $(\check\rho,\check\sigma,\check v)$ plane. We shall assume that the same feature holds for the partition function under consideration.

Since the quantization laws of Q^2, P^2 and $Q\cdot P$ imply that $\check\Phi(\check\rho,\check\sigma,\check v)$ is periodic under independent shifts of $\check\rho$, $\check\sigma$ and $\check v$ by 1, $1/r^2$ and $1/r$ respectively, eq.(2.1) can be inverted as

$$\begin{aligned} d(Q,P) \,=\, (-1)^{Q\cdot P+1} r^3 \int_{iM_1-1/2}^{iM_1+1/2} d\check\rho \int_{iM_2-1/(2r^2)}^{iM_2+1/(2r^2)} d\check\sigma \int_{iM_3-1/(2r)}^{iM_3+1/(2r)} d\check v \\ \times e^{-i\pi(\check\sigma Q^2+\check\rho P^2+2\check v Q\cdot P)}\, \frac{1}{\check\Phi(\check\rho,\check\sigma,\check v)} , \end{aligned} \tag{2.3}$$

provided the imaginary parts M_1, M_2 and M_3 of $\check\rho$, $\check\sigma$ and $\check v$ are fixed in a region where the original sum (2.1) is convergent.

Our proposal for $\check\Phi(\check\rho,\check\sigma,\check v)$ is

$$\check\Phi(\check\rho,\check\sigma,\check v)^{-1} = \sum_{\substack{s\in F,\, s|r \\ \bar s\equiv r/s}} g(s)\,\frac{1}{\bar s^3}\sum_{k=0}^{\bar s^2-1}\sum_{l=0}^{\bar s-1} \Phi_{10}\left(\check\rho, s^2\check\sigma+\frac{k}{\bar s^2}, s\check v+\frac{l}{\bar s}\right)^{-1} , \tag{2.4}$$

where

$$g(s) = s\,, \tag{2.5}$$

and $\Phi_{10}(\check\rho,\check\sigma,\check v)$ is the weight 10 Igusa cusp form of $\mathrm{Sp}(2,\mathbb{Z})$. The sum over k and l in (2.4) makes $\check\Phi$ periodic under $\check\sigma\to\check\sigma+(1/r^2)$ and $\check v\to\check v+(1/r)$ as required. Even though the function $g(s)$ has a simple form given in (2.5), we shall carry out our analysis keeping $g(s)$ arbitrary so that we can illustrate at the end how we fix the form of $g(s)$ from the known wall crossing formula for decay into a pair of primitive half-BPS dyons. In particular we shall show that across a wall of marginal stability associated with the decay of the original quarter BPS dyon into a pair of half BPS states carrying charges (Q_1,P_1) and (Q_2,P_2) with

(Q_1, P_1) being N_1 times a primitive lattice vector and (Q_2, P_2) being N_2 times a primitive lattice vector, the index jumps by an amount

$$\Delta d(Q,P)) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}(Q_1\cdot P_2 - Q_2\cdot P_1)\left\{\sum_{L_1|N_1} d_h\left(\frac{Q_1}{L_1},\frac{P_1}{L_1}\right)\right\}\left\{\sum_{L_2|N_2} d_h\left(\frac{Q_2}{L_2},\frac{P_2}{L_2}\right)\right\} \tag{2.6}$$

for the choice $g(s) = s$ in (2.4). Here $d_h(q,p)$ denotes the index of half BPS states carrying charges (q,p). When $N_1 = N_2 = 1$ both the decay products are primitive and (2.6) reduces to the standard wall crossing formula [13, 27−34].

Substituting (2.4) into (2.3), extending the ranges of $\check\sigma$ and $\check v$ integral with the help of the sums over k and l, and using the periodicity of Φ_{10} under integer shifts of its arguments we can get a simpler expression for the index:

$$\begin{aligned} d(Q,P) \;=\; & (-1)^{Q\cdot P+1}\sum_{s|r} g(s)\, s^3 \int_{iM_1-1/2}^{iM_1+1/2} d\check\rho \int_{iM_2-1/(2s^2)}^{iM_2+1/(2s^2)} d\check\sigma \int_{iM_3-1/(2s)}^{iM_3+1/(2s)} d\check v \\ & \times e^{-i\pi(\check\sigma Q^2+\check\rho P^2+2\check v Q\cdot P)}\, \Phi_{10}\left(\check\rho, s^2\check\sigma, s\check v\right)^{-1} . \end{aligned} \tag{2.7}$$

The set of dyons considered above contains only a subset of dyons of torsion r. This subset is known to be invariant under a $\Gamma^0(r)$ subgroup of the S-duality group [22]. This requires $\check\Phi$ to be invariant under the transformation [23]

$$\check\Phi(\check\rho',\check\sigma',\check v') = \check\Phi(\check\rho,\check\sigma,\check v) \qquad \text{for} \qquad \begin{pmatrix}\check\rho' & \check v'\\ \check v' & \check\sigma'\end{pmatrix} = \begin{pmatrix} d & b\\ c & a\end{pmatrix}\begin{pmatrix}\check\rho & \check v\\ \check v & \check\sigma\end{pmatrix}\begin{pmatrix} d & c\\ b & a\end{pmatrix},$$

$$a,c,d\in\mathbb{Z}, \qquad b\in r\mathbb{Z}, \qquad ad-bc=1\,. \tag{2.8}$$

On the other hand a general S-duality transformation matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ outside $\Gamma^0(r)$ will take us to dyons of torsion r outside the set $\mathcal{B}$ [22]. Thus with the help of these S-duality transformations on $\check\Phi$ we can determine the partition function for other torsion r dyons lying outside the set $\mathcal{B}$ considered above. In particular if we consider the set of dyons carrying charges (Q', P') related to (Q,P) via an S-duality transformation

$$(Q,P) = (aQ'+bP', cQ'+dP'), \quad \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \tag{2.9}$$

and denote by $1/\check\Phi'$ the partition function of these dyons, then $\check\Phi'$ is related to $\check\Phi$ via the relation

$$\check\Phi'(\check\rho',\check\sigma',\check v') = \check\Phi(\check\rho,\check\sigma,\check v) \quad \text{for} \quad \begin{pmatrix}\check\rho' & \check v'\\ \check v' & \check\sigma'\end{pmatrix} = \begin{pmatrix} d & b\\ c & a\end{pmatrix}\begin{pmatrix}\check\rho & \check v\\ \check v & \check\sigma\end{pmatrix}\begin{pmatrix} d & c\\ b & a\end{pmatrix}. \tag{2.10}$$

This allows us to determine the partition function of all other sets of torsion r dyons from the partition function given in (2.4). In particular for $r = 2$, choosing $\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}$ we recover the dyon partition function proposed in [23].

We shall now show that the proposed partition function (2.4) satisfies various consistency tests described in [23].

3. S-duality invariance

We shall first verify the required S-duality invariance of the partition function described in eq.(2.8). Using $\mathrm{Sp}(2,\mathbb{Z})$ invariance of $\Phi_{10}(x,y,z)$, and that b is a multiple of r for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(r)$ one can show that

$$\Phi_{10}\left(\check{\rho}', s^2\check{\sigma}' + \frac{k}{\bar{s}^2}, s\check{v}' + \frac{l}{\bar{s}}\right) = \Phi_{10}\left(\check{\rho}, s^2\check{\sigma} + \frac{k'}{\bar{s}^2}, s\check{v} + \frac{l'}{\bar{s}}\right), \tag{3.1}$$

where

$$k' = kd^2 - 2cdlr \in \mathbb{Z}, \qquad l' = (ad+bc)l - bdk/r \in \mathbb{Z}\,. \tag{3.2}$$

Thus

$$\sum_{k=0}^{\bar{s}^2-1}\sum_{l=0}^{\bar{s}-1}\Phi_{10}\left(\check{\rho}', s^2\check{\sigma}' + \frac{k}{\bar{s}^2}, s\check{v}' + \frac{l}{\bar{s}}\right)^{-1} = \sum_{k'=0}^{\bar{s}^2-1}\sum_{l'=0}^{\bar{s}-1}\Phi_{10}\left(\check{\rho}, s^2\check{\sigma} + \frac{k'}{\bar{s}^2}, s\check{v} + \frac{l'}{\bar{s}}\right)^{-1}, \tag{3.3}$$

and we have the required relation (2.8).

4. Wall crossing formula

We shall now verify that (2.7) is consistent with the wall crossing formula. As in [23] we shall only consider the decay into a pair of half-BPS dyons [13, 35 − 37]

$$(Q,P) \to (Q_1,P_1) + (Q_2,P_2)\,, \tag{4.1}$$

$$(Q_1,P_1) = (\alpha Q + \beta P, \gamma Q + \delta P), \qquad (Q_2,P_2) = (\delta Q - \beta P, -\gamma Q + \alpha P)\,, \tag{4.2}$$

$$\alpha\delta = \beta\gamma, \qquad \alpha + \delta = 1\,. \tag{4.3}$$

Since any lattice vector lying in the plane of Q and P can be expressed as a linear combination of Q/r and P with integer coefficients, we must have $\beta,\delta,\alpha \in \mathbb{Z}$, $\gamma \in \mathbb{Z}/r$. Thus we can write $\gamma = \gamma'/K$, where $K \in \mathbb{Z}$, $K|r$ and $\gcd(\gamma',K)$=1. The condition $\alpha\delta = \beta\gamma$ together with the integrality of α,β,δ now tells us that β must be of the form $K\beta'$ with $\beta' \in \mathbb{Z}$. Thus we have

$$\beta = K\,\beta', \quad \gamma = \frac{\gamma'}{K}, \quad K,\alpha,\delta,\beta',\gamma' \in \mathbb{Z}, \quad K|r, \quad \gcd(\gamma',K) = 1\,. \tag{4.4}$$

Using eqs.(4.3), (4.4) we have

$$\alpha + \delta = 1, \qquad \alpha\delta = \beta'\gamma', \qquad \alpha,\beta',\gamma',\delta \in \mathbb{Z}\,. \tag{4.5}$$

The analysis of [13] now shows that we can find a', b', c', d' such that

$$\alpha = a'd', \quad \beta' = -a'b', \quad \gamma' = c'd', \quad \delta = -b'c', \quad a',b',c',d' \in \mathbb{Z}, \quad a'd' - b'c' = 1\,. \tag{4.6}$$

As a consequence of (4.6) and the relation $\gcd(\gamma', K) = 1$ we have

$$\gcd(a', b') = \gcd(a', c') = \gcd(c', d') = \gcd(b', d') = 1, \quad \gcd(c', K) = \gcd(d', K) = 1\,. \tag{4.7}$$

Using eqs.(4.4)-(4.6) we can now express (4.2) as

$$(Q_1, P_1) = (a'K, c')(d'\bar{K}Q/r - b'P), \qquad (Q_2, P_2) = (b'K, d')(-c'\bar{K}Q/r + a'P), \tag{4.8}$$

$$\bar{K} \equiv r/K\,. \tag{4.9}$$

Since according to (4.7), $\gcd(a'K, c')$=1, $\gcd(b'K, d')$=1, and since any lattice vector lying in the Q-P plane can be expressed as integer linear combinations of Q/r and P, it follows from (4.8) that (Q_1, P_1) can be regarded as N_1 times a primitive vector and (Q_2, P_2) can be regarded as N_2 times a primitive vector where

$$N_1 = \gcd(d'\bar{K}, b') = \gcd(\bar{K}, b'), \qquad N_2 = \gcd(c'\bar{K}, a') = \gcd(\bar{K}, a')\,. \tag{4.10}$$

In the last steps we have again made use of (4.7). It follows from (4.7) and (4.10) that

$$\gcd(N_1, N_2) = 1, \quad N_1 N_2 | \bar{K}\,. \tag{4.11}$$

We shall now use the formula (2.7) for the index in different regions of the moduli space and calculate the change in the index as we cross the wall of marginal stability associated with the decay (4.1). For this we need to know how to choose the imaginary parts of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ along the integration contour for the two domains lying on the two sides of this wall of marginal stability. A prescription for choosing this contour was postulated in [23] according to which as we cross the wall of marginal stability associated with the decay (4.1), the integration contour crosses a pole of the partition function at

$$\check{\rho}\gamma - \check{\sigma}\beta + \check{v}(\alpha - \delta) = 0\,. \tag{4.12}$$

Thus the change in the index can be calculated by evaluating the residue of the partition function at this pole. We shall now examine for which values of s the $\Phi_{10}(\check{\rho}, s^2\check{\sigma}, s\check{v})^{-1}$ term in the expansion (2.7) has a pole at (4.12). The poles of $\Phi_{10}(\check{\rho}, s^2\check{\sigma}, s\check{v})^{-1}$ are known to be at

$$\begin{aligned} n_2\, s^2\,(\check{\rho}\check{\sigma} - \check{v}^2) + n_1 s^2 \check{\sigma} - m_1\check{\rho} + m_2 + js\check{v} &= 0 \\ m_1, n_1, m_2, n_2 \in \mathbb{Z}, \quad j \in 2\mathbb{Z}+1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} &= \frac{1}{4}\,. \end{aligned} \tag{4.13}$$

Comparing (4.13) and (4.12) we see that we must have

$$m_2 = n_2 = 0, \qquad j = \frac{\lambda}{s}(\alpha - \delta), \qquad n_1 = -\frac{\lambda}{s^2}\beta, \qquad m_1 = -\lambda\gamma\,, \tag{4.14}$$

for some λ. The last condition in (4.13), together with eqs.(4.3) now gives

$$\lambda = s\,. \tag{4.15}$$

Thus we have from (4.4), (4.14)

$$j = \alpha - \delta, \qquad m_1 = -s\gamma = -\gamma' s/K, \qquad n_1 = -\beta/s = -K\beta'/s\,. \tag{4.16}$$

Since $\gcd(\gamma', K) = 1$, the second equation in (4.16) shows that s must be a multiple of K:

$$s = K\,\tilde{s}, \qquad \tilde{s} \in \mathbb{Z}\,. \tag{4.17}$$

Substituting this into the last equation in (4.16) and using (4.6) we see that

$$n_1 = \frac{a'b'}{\tilde{s}} \quad \Rightarrow \quad \frac{a'b'}{\tilde{s}} \in \mathbb{Z}\,. \tag{4.18}$$

Since $\gcd(a', b')$=1, we must have a unique decomposition

$$\tilde{s} = L_1 L_2, \qquad L_1|b', \qquad L_2|a'\,. \tag{4.19}$$

On the other hand since s divides r, it follows from (4.17) that $\tilde{s}$ must divide $r/K = \bar{K}$. Thus

$$L_1|\bar{K}, \qquad L_2|\bar{K}\,. \tag{4.20}$$

It now follows from (4.10) that

$$L_1|N_1, \qquad L_2|N_2\,. \tag{4.21}$$

Conversely, given any pair (L_1, L_2) satisfying (4.21), it follows from (4.10) that L_1, L_2 will satisfy (4.19), (4.20). This allows us to find integers m_1, n_1, j satisfying (4.14) via eqs. (4.15)–(4.19).

This shows that the poles of $\check{\Phi}(\check{\rho}, \check{\sigma}, \check{v})^{-1}$ at (4.12) can come from the $s = KL_1L_2$ terms in (2.4) for $L_1|N_1$ and $L_2|N_2$. Our next task is to find the residues at these poles to compute the change in the index as we cross this wall. For this we define

$$\bar{a} = a'/L_2, \quad \bar{d} = d'L_2, \quad \bar{b} = b'/L_1, \quad \bar{c} = c'L_1, \quad s_0 = KL_1L_2\,. \tag{4.22}$$

It follows from (4.19) that $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are all integers. In terms of these variables the location of the pole given in (4.12) can be expressed as

$$s_0^{-1}\,\bar{c}\bar{d}\check{\rho} + \bar{a}\bar{b}s_0\check{\sigma} + (\bar{a}\bar{d} + \bar{b}\bar{c})\check{v} = 0\,. \tag{4.23}$$

We now define

$$\begin{aligned} \check{\rho}' &= \bar{d}^2\check{\rho} + \bar{b}^2 s_0^2\check{\sigma} + 2\bar{b}\bar{d}s_0\check{v}, \qquad\qquad \check{\sigma}' = \bar{c}^2\check{\rho} + \bar{a}^2 s_0^2\check{\sigma} + 2\bar{a}\bar{c}s_0\check{v}, \\ \check{v}' &= \bar{c}\bar{d}\check{\rho} + \bar{a}\bar{b}s_0^2\check{\sigma} + (\bar{a}\bar{d} + \bar{b}\bar{c})s_0\check{v}\,. \end{aligned} \tag{4.24}$$

The change of variables from $(\check{\rho}, s_0^2\check{\sigma}, s_0\check{v})$ to $(\check{\rho}', \check{\sigma}', \check{v}')$ is an $\mathrm{Sp}(2,\mathbb{Z})$ transformation. Thus we have

$$\Phi_{10}(\check{\rho}, s_0^2\check{\sigma}, s_0\check{v}) = \Phi_{10}(\check{\rho}', \check{\sigma}', \check{v}')\,. \tag{4.25}$$

In the primed variables the desired pole at (4.23) is at $\check{v}' = 0$. We also have

$$\check{\rho}P^2 + \check{\sigma}Q^2 + 2\check{v}Q\cdot P = \check{\rho}'P'^2 + \check{\sigma}'Q'^2 + 2\check{v}'Q'\cdot P'\,, \tag{4.26}$$

where

$$Q' = \bar{d}\,Q/s_0 - \bar{b}P, \qquad P' = -\bar{c}Q/s_0 + \bar{a}P\,. \tag{4.27}$$

Finally we have

$$d\check{\rho}' d\check{\sigma}' d\check{v}' = s_0^3 d\check{\rho} d\check{v} d\check{\sigma} \,. \tag{4.28}$$

This is consistent with the fact that $\Phi_{10}(\check{\rho}', \check{\sigma}', \check{v}')$ is invariant under integer shifts in $\check{\rho}'$, $\check{\sigma}'$ and $\check{v}'$ so that in the primed variables the volume of the unit cell is 1, while in the unprimed variables the volume of the unit cell is $1/s_0^3$. We can now express the change in the index from the $s = s_0$ term in (2.7) as

$$\begin{aligned}(\Delta d(Q,P))_{s_0} \;&=\; (-1)^{Q\cdot P+1}\, g(s_0) \int_{iM_1'-1/2}^{iM_1'+1/2} d\check{\rho}' \int_{iM_2'-1/2}^{iM_2'+1/2} d\check{\sigma}' \oint d\check{v}' \\ &\quad \times e^{-i\pi(\check{\sigma}' Q'^2 + \check{\rho}' P'^2 + 2\check{v}' Q'\cdot P')}\, \Phi_{10}\left(\check{\rho}', \check{\sigma}', \check{v}'\right)^{-1} \,,\end{aligned} \tag{4.29}$$

where the $\check{v}'$ contour is around the origin, — as in [23] we shall use the convention that the contour is in the clockwise direction. Using the fact that

$$\Phi_{10}\left(\check{\rho}', \check{\sigma}', \check{v}'\right) = -4\pi^2\, \check{v}'^2\, \eta(\check{\rho}')^{24}\, \eta(\check{\sigma}')^{24} + \mathcal{O}\left(\check{v}'^4\right) \,, \tag{4.30}$$

near $\check{v}' = 0$, we get

$$\begin{aligned}(\Delta d(Q,P))_{s_0} \;&=\; (-1)^{Q\cdot P+1}\, g(s_0)\, Q'\cdot P' \int_{iM_1'-1/2}^{iM_1'+1/2} d\check{\rho}' e^{-i\pi\check{\rho}' P'^2}\, \eta(\check{\rho}')^{-24} \\ &\qquad\qquad \times \int_{iM_2'-1/2}^{iM_2'+1/2} d\check{\sigma}' e^{-i\pi\check{\sigma}' Q'^2}\, \eta(\check{\sigma}')^{-24} \\ &=\; (-1)^{Q\cdot P+1}\, g(s_0)\, Q'\cdot P'\, d_h(Q',0)\, d_h(P',0) \,,\end{aligned} \tag{4.31}$$

where $d_h(q,0)$ denotes the index measuring the number of bosonic half BPS supermultiplets minus the number of fermionic half BPS supermultiplets carrying charge $(q,0)$.

We shall now express the right hand side of (4.31) in terms of the charges (Q_1, P_1) and (Q_2, P_2) of the decay products. First of all it is easy to see that

$$(-1)^{Q\cdot P} = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1} \,. \tag{4.32}$$

Furthermore it follows from (4.8), (4.22) and (4.27) that

$$(Q_1, P_1) = L_1\, \left(a' K Q', c' Q'\right) \,, \qquad (Q_2, P_2) = L_2\, \left(b' K P', d' P'\right) \,, \tag{4.33}$$

$$Q_1\cdot P_2 - Q_2\cdot P_1 = s_0\, Q'\cdot P' \,. \tag{4.34}$$

Now according to (4.7) the pair of integers $(a'K, c')$ are relatively prime, and the pair of integers $(b'K, d')$ are also relatively prime. Thus using S-duality invariance we can write

$$\begin{aligned} d_h\left(\frac{Q_1}{L_1}, \frac{P_1}{L_1}\right) &= d_h\left(a' K Q', c' Q'\right) = d_h(Q', 0), \\ d_h\left(\frac{Q_2}{L_2}, \frac{P_2}{L_2}\right) &= d_h\left(b' K P', d' P'\right) = d_h(P', 0) \,. \end{aligned} \tag{4.35}$$

Using (4.32), (4.34) and (4.35) we can now express (4.31) as

$$(\Delta d(Q,P))_{s_0} = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1}\, g(s_0)\, \frac{1}{s_0}\, (Q_1\cdot P_2 - Q_2\cdot P_1)\, d_h\left(\frac{Q_1}{L_1}, \frac{P_1}{L_1}\right)\, d_h\left(\frac{Q_2}{L_2}, \frac{P_2}{L_2}\right) \,. \tag{4.36}$$

For a given decay K is fixed but L_1 and L_2 can vary over all the factors of N_1 and N_2. Thus the total change in the index is obtained by summing over all possible values of s_0 of the form KL_1L_2. Thus gives

$$\begin{aligned}\Delta d(Q,P)) &= (-1)^{Q_1\cdot P_2-Q_2\cdot P_1+1}\,(Q_1\cdot P_2-Q_2\cdot P_1)\\ &\times\sum_{L_1|N_1}\sum_{L_2|N_2} g(KL_1L_2)\,\frac{1}{KL_1L_2}\,d_h\left(\frac{Q_1}{L_1},\frac{P_1}{L_1}\right)\,d_h\left(\frac{Q_2}{L_2},\frac{P_2}{L_2}\right).\end{aligned} \tag{4.37}$$

We can now fix the form of the function $g(s)$ by considering a decay where the decay products are primitive, i.e. $N_1 = N_2 = 1$. In this case we have $L_1 = L_2 = 1$, and (4.37) takes the form[1]

$$\Delta d(Q,P)) = (-1)^{Q_1\cdot P_2-Q_2\cdot P_1+1}\,(Q_1\cdot P_2-Q_2\cdot P_1)\,g(K)\,\frac{1}{K}\,d_h(Q_1,P_1)\,d_h(Q_2,P_2)\,. \tag{4.38}$$

In order that this agrees with the standard wall crossing formula for primitive decay [13, 27–33] we must have $g(K) = K$. Since this result should hold for all $K|r$ we see that we must set $g(s) = s$ as given in (2.5). Using this we can simplify the wall crossing formula (4.37) for generic non-primitive decays to the form given in (2.6).

5. Black hole entropy

In order to reproduce the leading contribution to the black hole entropy in the limit of large charges, the partition function must have a pole at [1]

$$\check\rho\check\sigma - \check v^2 + \check v = 0\,. \tag{5.1}$$

Furthermore, in order to reproduce the black hole entropy to first non-leading order, the inverse of the partition function near this pole must behave as [2, 23]

$$\check\Phi(\check\rho,\check\sigma,\check v) \propto (2v-\rho-\sigma)^{10}\,\left\{v^2\,\eta(\rho)^{24}\,\eta(\sigma)^{24} + \mathcal{O}\left(v^4\right)\right\}\,, \tag{5.2}$$

where

$$\rho = \frac{\check\rho\check\sigma-\check v^2}{\check\sigma},\quad \sigma = \frac{\check\rho\check\sigma-(\check v-1)^2}{\check\sigma},\quad v = \frac{\check\rho\check\sigma-\check v^2+\check v}{\check\sigma}\,. \tag{5.3}$$

[1] In this argument we have implicitly assumed that for a given K, it is possible to find integers a', b', c', d' satisfying $a'd' - b'c' = 1$, $\gcd(c', K) = \gcd(d', K) = \gcd(a', \bar K) = \gcd(b', \bar K) = 1$, so that (4.7) holds and we have $N_1 = N_2 = 1$ according to (4.10). If either K or $\bar K$ is odd then this assumption holds with the choice $\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 1 & 2\end{pmatrix}$. If K and $\bar K$ are both even then we cannot find a', b', c' and d' satisfying all the requirements since in order to satisfy $a'd' - b'c' = 1$ at least one of them must be even. However in this case if we choose $\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}$ then we satisfy (4.7) and have $N_1 = 1$ and $N_2 = 2$ according to (4.10). We can now demand that the wall crossing formula in this case agrees with the one derived in [23] for decays where one of the decay products is twice a primitive vector. This gives $g(K) = K$ even for K, $\bar K$ both even.

We shall now examine the poles of $\check{\Phi}$ given in (2.4) to see if it satisfies the above relations. For this we recall eq.(4.13) giving the pole of $\Phi_{10}(\check{\rho}, s^2\check{\sigma}, s\check{v})^{-1}$. Comparing (4.13) with (5.1) we see that in order to get a pole at (5.1) we must choose in (4.13)

$$n_2 = \lambda/s^2, \qquad j = \lambda/s, \qquad n_1 = m_1 = m_2 = 0\,, \tag{5.4}$$

for some λ. The requirement $m_1 n_1 + m_2 n_2 + \frac{1}{4}j^2 = \frac{1}{4}$ then gives

$$\lambda = s, \qquad n_2 = \frac{1}{s}\,. \tag{5.5}$$

Since n_2 must be an integer this gives $s = 1$. Thus only the $s = 1$ term in (2.4) has a pole at (5.1). It follows from the known behaviour of Φ_{10} near its zeroes that $\check{\Phi}$ defined in (2.4) satisfies the requirement (5.2).

6. Gauge theory limit

Finally we shall show that in special regions of the Narain moduli space where the low lying states in string theory describe a non-abelian gauge theory, the proposed dyon spectrum of string theory reproduces the known results in gauge theory. Since the T-duality invariant metric L in the Narain moduli space descends to the negative of the Cartan metric in gauge theories, and since the Cartan metric is positive definite, all the gauge theory dyons have the property

$$Q^2 < 0, \qquad P^2 < 0, \qquad Q^2P^2 > (Q\cdot P)^2\,. \tag{6.1}$$

Thus in order to identify dyons which could be interpreted as gauge theory dyons in an appropriate limit we must focus on charge vectors satisfying (6.1).

In order to identify such dyons we need to expand the partition function $\check{\Phi}^{-1}$ in powers of $e^{2\pi i\check{\rho}}$, $e^{2\pi i\check{\sigma}}$ and $e^{2\pi i\check{v}}$ and pick up the appropriate terms in the expansion. For this we need to identify a region in the $(\check{\rho}, \check{\sigma}, \check{v})$ space (or equivalently in the asymptotic moduli space) where we carry out the expansion, since in different regions we have different expansion. We shall consider the region

$$\Im(\check{\rho}), \Im(\check{\sigma}) \gg -\Im(\check{v}) \gg 1\,, \tag{6.2}$$

where $\Im(z)$ denotes the imaginary part of z. The results in other relevant regions will be related to the ones in this region by S-duality transformation. In the region (6.2) the only term in $\Phi_{10}\left(\check{\rho}, s^2\check{\sigma} + \frac{k}{s^2}, s\check{v} + \frac{l}{s}\right)^{-1}$ which has powers of $e^{2\pi i\check{\rho}}$, $e^{2\pi i\check{\sigma}}$ and $e^{2\pi i\check{v}}$ compatible with the requirement (6.1) is

$$e^{-2\pi i\check{\rho} - 2\pi i(s^2\check{\sigma} + \frac{k}{s^2}) - 2\pi i(s\check{v} + \frac{l}{s})}\,. \tag{6.3}$$

This is in fact the leading term in the expansion in the limit (6.2). Substituting this into (2.4) and performing the sum over k and l we see that only the $s = r$ term in the sum survives and the result is

$$r\, e^{-2\pi i(\check{\rho} + r^2\check{\sigma} + r\check{v})}\,. \tag{6.4}$$

This corresponds to dyons with

$$Q^2/2 = -r^2, \quad P^2/2 = -1, \quad Q \cdot P = -r\,, \tag{6.5}$$

with an index of $(-1)^{r+1}\, r$.

We can also determine the walls of marginal stability which border the domain in which these dyons exist. This requires determining the region in the $(\Im(\check\rho), \Im(\check\sigma), \Im(\check v))$ space in which the expansion (6.4) of $\Phi_{10}(\check\rho, r^2\check\sigma, r\check v)^{-1}$ is valid since then we can determine the associated walls of marginal stability using (4.12). For this we shall utilize the known results for $r = 1$; in this case the walls of marginal stability bordering the domain in which (6.4) is valid correspond to the decays into $(Q, 0) + (0, P)$, $(Q - P, 0) + (P, P)$ and $(0, P - Q) + (Q, Q)$ respectively [38]. Using (4.12) we now see that validity of the expansion of $\Phi_{10}(\check\rho, \check\sigma, \check v)^{-1}$ given by (6.4) with $r = 1$ is bounded by the following surfaces in the $(\Im(\check\rho), \Im(\check\sigma), \Im(\check v))$ space:

$$\Im(\check v) = 0\,, \qquad \Im(\check v + \check\sigma) = 0, \qquad \Im(\check v + \check\rho) = 0\,. \tag{6.6}$$

We can now simply scale $\check\sigma$ by r^2 and $\check v$ by r to determine the region of validity of the expansion (6.4) for $\Phi_{10}(\check\rho, r^2\check\sigma, r\check v)$:

$$\Im(\check v) = 0\,, \qquad \Im(\check v + r\check\sigma) = 0, \qquad \Im(r\check v + \check\rho) = 0\,. \tag{6.7}$$

Comparing these with (4.12) we now see that the corresponding walls of marginal stability are associated with the decays into

$$(Q, 0) + (0, P), \qquad (Q - rP, 0) + (rP, P), \qquad \left(0, P - \frac{1}{r}Q\right) + \left(Q, \frac{1}{r}Q\right)\,. \tag{6.8}$$

Let us now compare these results with dyons in $\mathcal{N} = 4$ supersymmetric SU(3) gauge theory. If we denote by α_1 and α_2 a pair of simple roots of SU(3) with $\alpha_1^2 = \alpha_2^2 = -2$ and $\alpha_1 \cdot \alpha_2 = 1$, then the analysis of [39 − 43] shows that the gauge theory contains dyons of charge

$$(Q, P) = (r\alpha_1, -\alpha_2)\,, \tag{6.9}$$

with index $(-1)^{r+1}\, r$. These are precisely the dyons of the type given in (6.5). Furthermore using string junction picture [44, 45], ref. [39] also determined the walls of marginal stability bordering the domain in which these dyons exist. These also coincide with (6.8).

The spectrum in gauge theory contains other dyons of torsion r related to the ones described above by S-duality transformation. Since our construction is manifestly S-duality invariant, the results for these dyons can also be reproduced from the general formula given in (2.4).

Gauge theory also contains other dyons which are not related to the ones described here by S-duality [41, 42]. These typically require higher gauge groups and has additional fermionic zero modes besides the ones required by broken supersymmetry. Quantization of these additional zero modes gives rise to additional bose-fermi degeneracy, and as a result the index being computed here vanishes for these dyons. This is also apparent from the fact that these dyons typically exist only in a subspace of the full moduli space; as a result when we move away from this subspace the various states combine and become non-BPS. Some aspects of these dyons have been discussed recently in [46, 43].

Acknowledgments

We would like to thank Nabamita Banerjee, Miranda Cheng, Atish Dabholkar, Justin David, Frederik Denef, Dileep Jatkar, Suresh Nampuri, K. Narayan and Sunil Mukhi for useful discussions.

References

[1] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Counting dyons in $N = 4$ string theory*, *Nucl. Phys.* **B 484** (1997) 543 [hep-th/9607026].

[2] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [hep-th/0412287].

[3] D. Shih, A. Strominger and X. Yin, *Recounting dyons in $N = 4$ string theory*, *JHEP* **10** (2006) 087 [hep-th/0505094].

[4] D. Gaiotto, *Re-recounting dyons in $N = 4$ string theory*, hep-th/0506249.

[5] D. Shih and X. Yin, *Exact black hole degeneracies and the topological string*, *JHEP* **04** (2006) 034 [hep-th/0508174].

[6] D.P. Jatkar and A. Sen, *Dyon spectrum in CHL models*, *JHEP* **04** (2006) 018 [hep-th/0510147].

[7] J.R. David, D.P. Jatkar and A. Sen, *Product representation of dyon partition function in CHL models*, *JHEP* **06** (2006) 064 [hep-th/0602254].

[8] A. Dabholkar and S. Nampuri, *Spectrum of dyons and black holes in CHL orbifolds using Borcherds lift*, *JHEP* **11** (2007) 077 [hep-th/0603066].

[9] J.R. David and A. Sen, *CHL dyons and statistical entropy function from $D1$-$D5$ system*, *JHEP* **11** (2006) 072 [hep-th/0605210].

[10] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in $N = 4$ supersymmetric type-II string theories*, *JHEP* **11** (2006) 073 [hep-th/0607155].

[11] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in generic $N = 4$ supersymmetric $Z(N)$ orbifolds*, *JHEP* **01** (2007) 016 [hep-th/0609109].

[12] A. Dabholkar and D. Gaiotto, *Spectrum of CHL dyons from genus-two partition function*, *JHEP* **12** (2007) 087 [hep-th/0612011].

[13] A. Sen, *Walls of marginal stability and dyon spectrum in $N = 4$ supersymmetric string theories*, *JHEP* **05** (2007) 039 [hep-th/0702141].

[14] A. Dabholkar, D. Gaiotto and S. Nampuri, *Comments on the spectrum of CHL dyons*, *JHEP* **01** (2008) 023 [hep-th/0702150].

[15] N. Banerjee, D.P. Jatkar and A. Sen, *Adding charges to $N = 4$ dyons*, *JHEP* **07** (2007) 024 [arXiv:0705.1433].

[16] A. Sen, *Two centered black holes and $N = 4$ dyon spectrum*, *JHEP* **09** (2007) 045 [arXiv:0705.3874].

[17] M.C.N. Cheng and E. Verlinde, *Dying dyons don't count*, *JHEP* **09** (2007) 070 [arXiv:0706.2363].

[18] A. Sen, *Black hole entropy function, attractors and precision counting of microstates*, arXiv:0708.1270.

[19] D. Shih, A. Strominger and X. Yin, *Counting dyons in $N = 8$ string theory*, *JHEP* **06** (2006) 037 [hep-th/0506151].

[20] J.R. David, *On the dyon partition function in $N = 2$ theories*, *JHEP* **02** (2008) 025 [arXiv:0711.1971].

[21] S. Banerjee and A. Sen, *Duality orbits, dyon spectrum and gauge theory limit of heterotic string theory on T^6*, *JHEP* **03** (2008) 022 [arXiv:0712.0043].

[22] S. Banerjee and A. Sen, *S-duality action on discrete T-duality invariants*, *JHEP* **04** (2008) 012 [arXiv:0801.0149].

[23] S. Banerjee, A. Sen and Y.K. Srivastava, *Generalities of quarter BPS dyon partition function and dyons of torsion two*, arXiv:0802.0544.

[24] K.S. Narain, *New heterotic string theories in uncompactified dimensions < 10*, *Phys. Lett.* **B 169** (1986) 41.

[25] K.S. Narain, M.H. Sarmadi and E. Witten, *A note on toroidal compactification of heterotic string theory*, *Nucl. Phys.* **B 279** (1987) 369.

[26] A. Gregori et al., *R^2 corrections and non-perturbative dualities of $N = 4$ string ground states*, *Nucl. Phys.* **B 510** (1998) 423 [hep-th/9708062].

[27] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **08** (2000) 050 [hep-th/0005049].

[28] F. Denef, *On the correspondence between D-branes and stationary supergravity solutions of type-II Calabi-Yau compactifications*, hep-th/0010222.

[29] F. Denef, B.R. Greene and M. Raugas, *Split attractor flows and the spectrum of BPS D-branes on the quintic*, *JHEP* **05** (2001) 012 [hep-th/0101135].

[30] F. Denef, *Quantum quivers and Hall/hole halos*, *JHEP* **10** (2002) 023 [hep-th/0206072].

[31] B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, hep-th/0304094.

[32] F. Denef and G.W. Moore, *Split states, entropy enigmas, holes and halos*, hep-th/0702146.

[33] E. Diaconescu and G.W. Moore, *Crossing the wall: branes vs. bundles*, arXiv:0706.3193.

[34] M. Kontsevich, *Stability, wall-crossing and counting of objects in Calabi-Yau categories*, talk given at the $XXXVII^{\text{th}}$ *Paris summer institute: black holes, black rings and modular forms*, Paris France (2007).

[35] A. Sen, *Rare decay modes of quarter BPS dyons*, *JHEP* **10** (2007) 059 [arXiv:0707.1563].

[36] A. Mukherjee, S. Mukhi and R. Nigam, *Dyon death eaters*, *JHEP* **10** (2007) 037 [arXiv:0707.3035].

[37] A. Mukherjee, S. Mukhi and R. Nigam, *Kinematical analogy for marginal dyon decay*, arXiv:0710.4533.

[38] A. Sen, *Three string junction and $N = 4$ dyon spectrum*, *JHEP* **12** (2007) 019 [arXiv:0708.3715].

[39] K.-M. Lee and P. Yi, *Dyons in $N = 4$ supersymmetric theories and three-pronged strings*, *Phys. Rev.* **D 58** (1998) 066005 [hep-th/9804174].

[40] D. Bak, K.-M. Lee and P. Yi, *Quantum* 1/4 *BPS dyons*, *Phys. Rev.* **D 61** (2000) 045003 [hep-th/9907090].

[41] M. Stern and P. Yi, *Counting Yang-Mills dyons with index theorems*, *Phys. Rev.* **D 62** (2000) 125006 [hep-th/0005275].

[42] E.J. Weinberg and P. Yi, *Magnetic monopole dynamics, supersymmetry and duality*, *Phys. Rept.* **438** (2007) 65 [hep-th/0609055].

[43] A. Dabholkar, K. Narayan and S. Nampuri, *Degeneracy of decadent dyons*, *JHEP* **03** (2008) 026 [arXiv:0802.0761].

[44] O. Bergman, *Three-pronged strings and* 1/4 *BPS states in $N = 4$ super-Yang-Mills theory*, *Nucl. Phys.* **B 525** (1998) 104 [hep-th/9712211].

[45] O. Bergman and B. Kol, *String webs and* 1/4 *BPS monopoles*, *Nucl. Phys.* **B 536** (1998) 149 [hep-th/9804160].

[46] K. Narayan, *On the internal structure of dyons in $\mathcal{N} = 4$ super Yang-Mills theories*, *Phys. Rev.* **D 77** (2008) 046004 [arXiv:0712.3625].